D-brane probes on $L^{a,b,c}$ superconformal field theories

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ABSTRACT: We study supersymmetric embeddings of D-brane probes of different dimensionality in the $AdS_5 \times L^{a,b,c}$ background of type IIB string theory. In the case of D3-branes, we recover the known three-cycles dual to the dibaryonic operators of the gauge theory and we also find a new family of supersymmetric embeddings. Supersymmetric configurations of D5-branes, representing fractional branes, and of spacetime filling D7-branes (which can be used to add flavor) are also found. Stable non supersymmetric configurations corresponding to fat strings and domain walls are found as well.

KEYWORDS: Gauge-gravity correspondence, D-branes, Brane Dynamics in Gauge Theories, AdS-CFT Correspondence.

Contents

1. Introduction

The study of supersymmetric D–brane probes in a given background is relevant to extract *stringy* information in the framework of gauge/gravity duality. This is a well-known fact since Witten early showed, in the case of $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) theory, that the gravity side must contain branes in order to accommodate the Pfaffian operator –in the SO(2N) case–, as well as the baryon vertex or domain walls arising in SU(N) gauge theories [1]. Moreover, the introduction of flavor in the gauge theory side forces to consider an open string sector in the dual theory [2]. As a consequence, any theory in the universality class of QCD demands a clear understanding of these features.

In the long path from Maldacena's original setup to more realistic scenarios, an important framework has been considered in recent years. If the five-sphere of the background is replaced by any Sasaki–Einstein five-dimensional manifold X^5, a duality between type IIB string theory on $AdS_5 \times X^5$ and a superconformal quiver gauge theory arises [3]. Until two years ago, the only case at hand whose complete metric was known was $X^5 = T^{1,1}$, that leads to the so-called Klebanov–Witten model [4]. More recently, two new families

of infinitely many Sasaki–Einstein manifolds were built and their metrics were explicitly constructed. They are labeled by two positive integers $Y^{p,q}$ [5] or three positive integers $L^{a,b,c}$ [6, 7]. Indeed, the former can be seen as a subfamily of the latter. The corresponding superconformal field theories were constructed almost immediately in, respectively, [8, 9] and [10 – 12], by exploiting the rich mathematical structure of toric geometry. These families exhaust all possible toric Calabi–Yau cones on a base with topology $S^2 \times S^3$. Research on AdS/CFT in these superconformal gauge theories has led to a better understanding of several important issues such as the appearance of duality cascades, a–maximization, Seiberg duality, etc.

In order to determine the supersymmetric embeddings of D–brane probes we employ kappa symmetry [13]. Our approach is based on the existence of a matrix Γ_κ which depends on the metric induced on the worldvolume of the probe and characterizes its supersymmetric embeddings. If ϵ is a Killing spinor of the background, only those embeddings such that $\Gamma_\kappa \epsilon = \epsilon$ preserve a fraction of the background supersymmetry [14]. This condition gives rise to a set of first-order BPS differential equations whose solutions determine the details of the embedding. As well, they solve the equations of motion derived from the DBI action of the probe while saturating a bound for the energy, as it usually happens in the case of worldvolume solitons [15].

D-brane probes in the Klebanov–Witten model were studied in full detail in [16]. In the case of $Y^{p,q}$ superconformal gauge theories, the exhaustive research was undertaken more recently in [17]. These articles explore interesting features such as excitations of dibaryons, the baryon vertex, the presence of domain walls, fat strings and defect conformal field theories in the quiver theory side. In this letter, we aim at filling the gap by giving the main results in the case of $L^{a,b,c}$ theories.

The content of this article is organized as follows. In section 2 we review those aspects of the $L^{a,b,c}$ spaces that we need afterwards. Section 3 deals with the construction of local complex coordinates and other geometrical aspects of the Calabi–Yau cone, $\mathcal{C}L^{a,b,c}$. In section 4 we provide the expression for the Killing spinors on $AdS_5 \times L^{a,b,c}$. We briefly describe the basics of the dual superconformal quiver theories in section 5. We consider D3–branes wrapping supersymmetric 3-cycles dual to dibaryonic operators in section 6. Besides matching their quantum numbers, we find general holomorphic embeddings corresponding to divisors of $\mathcal{C}L^{a,b,c}$. In section 7 we consider D5–branes with the focus on fractional branes, while section 8 deals with spacetime filling configurations of D7–branes that can be used to introduce flavor. We finally comment on some stable non-supersymmetric configurations representing fat strings and domain walls in section 9, where we furthermore present our conclusions.

2. The $L^{a,b,c}$ geometry

The Sasaki–Einstein manifold $L^{a,b,c}$ is a five-dimensional space with topology $S^2 \times S^3$, whose metric can be written as [6]:

$$ds^2_{L^{a,b,c}} = ds^2_4 + (d\tau + \sigma)^2 \,, \tag{2.1}$$

where ds_4^2 is a local Kähler–Einstein metric, with Kähler form $J_4 = \frac{1}{2}d\sigma$, given by

$$ds_4^2 = \frac{\rho^2}{4\Delta_x}\,dx^2 + \frac{\rho^2}{\Delta_\theta}\,d\theta^2 + \frac{\Delta_x}{\rho^2}\left(\frac{\sin^2\theta}{\alpha}\,d\phi + \frac{\cos^2\theta}{\beta}\,d\psi\right)^2 +$$
$$+ \frac{\Delta_\theta\sin^2\theta\cos^2\theta}{\rho^2}\left[\left(1-\frac{x}{\alpha}\right)d\phi - \left(1-\frac{x}{\beta}\right)d\psi\right]^2, \qquad (2.2)$$

the quantities Δ_x, Δ_θ, ρ^2 and σ reading

$$\Delta_x = x(\alpha - x)(\beta - x) - \mu\,,$$
$$\Delta_\theta = \alpha\cos^2\theta + \beta\sin^2\theta\,, \qquad \rho^2 = \Delta_\theta - x\,,$$
$$\sigma = \left(1-\frac{x}{\alpha}\right)\sin^2\theta\,d\phi + \left(1-\frac{x}{\beta}\right)\cos^2\theta\,d\psi\,. \qquad (2.3)$$

The ranges of the different coordinates are $0 \le \theta \le \pi/2$, $x_1 \le x \le x_2$, $0 \le \phi, \psi < 2\pi$, where x_1 and x_2 are the smallest roots of the cubic equation $\Delta_x = 0$. A natural tetrad frame for this space reads

$$e^1 = \frac{\rho}{\sqrt{\Delta_\theta}}\,d\theta\,, \qquad e^2 = \frac{\sqrt{\Delta_\theta}\,\sin\theta\cos\theta}{\rho}\left(\left(1-\frac{x}{\alpha}\right)d\phi - \left(1-\frac{x}{\beta}\right)d\psi\right),$$
$$e^3 = \frac{\sqrt{\Delta_x}}{\rho}\left(\frac{\sin^2\theta}{\alpha}\,d\phi + \frac{\cos^2\theta}{\beta}\,d\psi\right),$$
$$e^4 = \frac{\rho}{2\sqrt{\Delta_x}}\,dx\,, \qquad e^5 = (\,d\tau + \sigma\,)\,. \qquad (2.4)$$

Notice that, in this frame, $J_4 = e^1 \wedge e^2 + e^3 \wedge e^4$. Let us now define a_i, b_i and c_i ($i = 1, 2$) as follows:

$$a_i = \frac{\alpha c_i}{x_i - \alpha}\,, \qquad b_i = \frac{\beta c_i}{x_i - \beta}\,, \qquad c_i = \frac{(\alpha - x_i)(\beta - x_i)}{2(\alpha + \beta)\,x_i - \alpha\beta - 3x_i^2}\,. \qquad (2.5)$$

The coordinate τ happens to be compact and varies between 0 and $\Delta\tau$,

$$\Delta\tau = \frac{2\pi k|c_1|}{b}\,, \qquad k = \gcd\,(a, b)\,. \qquad (2.6)$$

The a_i, b_i and c_i constants are related to the integers a, b, c of $L^{a,b,c}$ by means of the relations:

$$a\,a_1 + b\,a_2 + c = 0\,, \qquad a\,b_1 + b\,b_2 + d = 0\,, \qquad a\,c_1 + b\,c_2 = 0\,, \qquad (2.7)$$

where $d = a + b - c$. The constants α, β and μ appearing in the metric are related to the roots x_1, x_2 and x_3 of Δ_x as

$$\mu = x_1 x_2 x_3\,, \qquad \alpha + \beta = x_1 + x_2 + x_3\,, \qquad \alpha\beta = x_1 x_2 + x_1 x_3 + x_2 x_3\,. \qquad (2.8)$$

Moreover, it follows from (2.7) that all ratios bewteen the four quantities $a_1 c_2 - a_2 c_1$, $b_1 c_2 - b_2 c_1$, c_1, and c_2 must be rational. Actually, one can prove that:

$$\frac{a_1 c_2 - a_2 c_1}{c_1} = \frac{c}{b}, \qquad \frac{b_1 c_2 - b_2 c_1}{c_1} = \frac{d}{b}, \qquad \frac{c_1}{c_2} = -\frac{b}{a}. \qquad (2.9)$$

Any other ratio between (a, b, c, d) can be obtained by combining these equations. In particular, from (2.5), (2.8) and (2.9), one can rewrite some of these relations as:

$$\frac{a}{b} = \frac{x_1}{x_2} \frac{x_3 - x_1}{x_3 - x_2}, \qquad \frac{a}{c} = \frac{(\alpha - x_2)(x_3 - x_1)}{\alpha(\beta - x_1)},$$

$$\frac{c}{d} = \frac{\alpha}{\beta} \frac{(\beta - x_1)(\beta - x_2)}{(\alpha - x_1)(\alpha - x_2)} = \frac{\alpha}{\beta} \frac{x_3 - \alpha}{x_3 - \beta}. \qquad (2.10)$$

The manifold has $U(1) \times U(1) \times U(1)$ isometry. It is, thus, toric. Its volume can be computed from the metric with the result:

$$\mathrm{Vol}(L^{a,b,c}) = \frac{(x_2 - x_1)(\alpha + \beta - x_1 - x_2)\,|c_1|}{\alpha \beta b}\,\pi^3. \qquad (2.11)$$

Other geometrical aspects of these spaces can be found in [6, 8].

3. Complex coordinates on $\mathcal{CL}^{a,b,c}$

In order to construct a set of local complex coordinates on the Calabi–Yau cone on $L^{a,b,c}$, $\mathcal{CL}^{a,b,c}$, let us introduce the following basis of closed one-forms [1]

$$\hat{\eta}_1 = \alpha \frac{\cot \theta}{\Delta_\theta}\, d\theta - \frac{\alpha(\beta - x)}{2\Delta_x}\, dx + i d\phi,$$

$$\hat{\eta}_2 = -\beta \frac{\tan \theta}{\Delta_\theta}\, d\theta - \frac{\beta(\alpha - x)}{2\Delta_x}\, dx + i d\psi,$$

$$\hat{\eta}_3 = \frac{dr}{r} + i d\tau + (\beta - \alpha)\frac{\sin(2\theta)}{2\Delta_\theta}\, d\theta + \frac{(\alpha - x)(\beta - x)}{2\Delta_x}\, dx. \qquad (3.1)$$

From these quantities, it is possible to define a set of $(1,0)$-forms η_i as the following linear combinations:

$$\eta_1 = \hat{\eta}_1 - \hat{\eta}_2, \qquad \eta_2 = \hat{\eta}_1 + \hat{\eta}_2, \qquad \eta_3 = 3\hat{\eta}_3 + \hat{\eta}_1 + \hat{\eta}_2. \qquad (3.2)$$

One can immediately check that they are integrable, $\eta^i = \frac{dz^i}{z^i}$. The explicit form of the complex coordinates z^i is:

$$z_1 = \tan \theta\, f_1(x)\, e^{i(\phi - \psi)}, \qquad z_2 = \frac{\sin(2\theta)}{f_2(x)\,\Delta_\theta}\, e^{i(\phi + \psi)},$$

$$z_3 = r^3\, \sin(2\theta)\, \sqrt{\Delta_\theta \Delta_x}\, e^{i(3\tau + \phi + \psi)}, \qquad (3.3)$$

where

$$f_1(x) = \mathcal{P}_1(x)^{\alpha - \beta}, \qquad f_2(x) = \mathcal{P}_0(x)^{2\,\alpha\,\beta}\, \mathcal{P}_1(x)^{-(\alpha + \beta)}, \qquad (3.4)$$

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[1]Notice that there are a few sign differences in our conventions as compared to those in [18].

and the functions $\mathcal{P}_q(x)$ are defined as

$$\mathcal{P}_q(x) = \exp\left(\int \frac{x^q\, dx}{2\,\Delta_x}\right) = \prod_{i=1}^{3} (x - x_i)^{\frac{1}{2}\frac{x_i^q}{\Pi_{j\neq i}^{3}\,(x_i - x_j)}}\,. \tag{3.5}$$

In terms of these $(1,0)$-forms, it is now fairly simple to work out the two-form Ω_4,

$$\Omega_4 = 3e^{i(\phi+\psi)}\,\sin(2\theta)\sqrt{\Delta_\theta \Delta_x}\,\eta_1 \wedge \eta_2\,, \tag{3.6}$$

obeying $d\,\Omega_4 = 3i\sigma \wedge \Omega_4$. By using these properties one can verify that the three-form:

$$\Omega = r^2\, e^{3i\tau}\, \Omega_4 \wedge [\,dr + ir\,(d\tau + \sigma)\,]\,, \tag{3.7}$$

is closed. Moreover, the explicit expression for Ω in terms of the above defined closed and integrable $(1,0)$-forms reads

$$\Omega = r^3\,\sin(2\theta)\,e^{i(3\tau+\phi+\psi)}\,\sqrt{\Delta_\theta \Delta_x}\,\eta_1 \wedge \eta_2 \wedge \eta_3\,, \tag{3.8}$$

which shows that $\Omega \wedge \eta_i = 0$. In terms of the complex coordinates z_i, the form Ω adopts a simple expression from which it is clear that it is the holomorphic $(3,0)$ form of the Calabi-Yau cone $\mathcal{C}L^{a,b,c}$,

$$\Omega = \frac{dz_1 \wedge dz_2 \wedge dz_3}{z_1 z_2}\,. \tag{3.9}$$

The expression (3.8) allows for the right identification of the angle conjugated to the R–symmetry [10],

$$\psi' = 3\tau + \phi + \psi\,. \tag{3.10}$$

Finally, starting from J_4, we can write the Kähler form J of $\mathcal{C}L^{a,b,c}$,

$$J = r^2\,J_4 + r\,dr \wedge e^5\,, \qquad dJ = 0\,. \tag{3.11}$$

Notice that all the expressions written in this section reduce to those of $\mathcal{C}Y^{p,q}$ provided

$$\begin{aligned}
a &= p - q\,, \qquad b = p + q\,, \qquad c = p\,, \\
3x - \alpha &= 2\,\alpha\,y\,, \qquad \mu = \frac{4}{27}\,(1-a)\,\alpha^3\,, \\
\tilde{\theta} &= 2\theta\,, \qquad \tilde{\beta} = -(\phi+\psi)\,, \qquad \tilde{\phi} = \phi - \psi\,,
\end{aligned} \tag{3.12}$$

while (3.10) provides the right identification with the $U(1)_R$ angle in $Y^{p,q}$. We shall use this limiting case several times along the letter to make contact with the results found in [17].

4. Killing spinors for $AdS_5 \times L^{a,b,c}$

In order to study D–brane probes' embeddings by means of kappa symmetry, we need to know the Killing spinors of the string theory background. The solution of type IIB supergravity corresponding to the near-horizon region of a stack of N coincident D3-branes

located at the apex of the $\mathcal{C}L^{a,b,c}$ cone, is characterized by a ten-dimensional metric,

$$ds^2 = \frac{r^2}{L^2}\, dx_{1,3}^2 + \frac{L^2}{r^2}\, dr^2 + L^2\, ds^2_{L^{a,b,c}}\,, \tag{4.1}$$

and a self-dual Ramond-Ramond five-form $F^{(5)}$ given by:

$$g_s\, F^{(5)} = d^4x \wedge dh^{-1} + \text{Hodge dual}\,, \qquad h(r) = \frac{L^4}{r^4}\,. \tag{4.2}$$

The quantization condition of the flux of $F^{(5)}$ determines the constant L in terms of g_s, N, α' and the volume of the Sasaki–Einstein space:

$$L^4 = \frac{4\pi^4}{\text{Vol}(L^{a,b,c})}\, g_s\, N\, (\alpha')^2\,. \tag{4.3}$$

The Killing spinors of the $AdS_5 \times L^{a,b,c}$ background can be written in terms of a constant spinor η,

$$\epsilon = e^{\frac{i}{2}(3\tau+\phi+\psi)}\, r^{-\frac{\Gamma_*}{2}} \left(1 + \frac{\Gamma_r}{2L^2}\, x^\alpha\, \Gamma_{x^\alpha}\, (1+\Gamma_*) \right) \eta\,, \tag{4.4}$$

where we have introduced the matrix $\Gamma_* \equiv i\Gamma_{x^0x^1x^2x^3}$. The spinor η satisfies the projections [18]:

$$\Gamma_{12}\,\eta = i\eta\,, \qquad \Gamma_{34}\,\eta = i\eta\,, \tag{4.5}$$

this implying that ϵ also satisfies the same projections. It is convenient to decompose the constant spinor η according to its Γ_*–parity, $\Gamma_*\,\eta_\pm = \pm\eta_\pm$. Using this decomposition, we obtain two types of Killing spinors:

$$\begin{aligned}
e^{-\frac{i}{2}(3\tau+\phi+\psi)}\, \epsilon_- &= r^{1/2}\,\eta_-\,, \\
e^{-\frac{i}{2}(3\tau+\phi+\psi)}\, \epsilon_+ &= r^{-1/2}\,\eta_+ + \frac{r^{1/2}}{L^2}\,\Gamma_r\, x^\alpha\, \Gamma_{x^\alpha}\,\eta_+\,.
\end{aligned} \tag{4.6}$$

The spinors ϵ_- satisfy $\Gamma_*\,\epsilon_- = -\epsilon_-$, whereas the ϵ_+'s are not eigenvectors of Γ_*. The former correspond to ordinary supercharges while the latter, which depend on the x^α coordinates, are related to the superconformal supersymmetries. The only dependence on the coordinates of $L^{a,b,c}$ is through the exponential of $\psi' = 3\tau + \phi + \psi$. This angle, as explained above, is identified with the $U(1)_R$ of the superconformal quiver theory.

It is finally convenient to present the explicit expression for the Killing spinors when AdS_5 is described by its global coordinates,

$$ds^2_{AdS_5} = L^2 \left[-\cosh^2\varrho\, dt^2 + d\varrho^2 + \sinh^2\varrho\, d\Omega_3^2 \right]\,, \tag{4.7}$$

where $d\Omega_3^2$ is the round metric of a unit three-sphere. Let us parameterize $d\Omega_3^2$ in terms of three angles $(\alpha^1, \alpha^2, \alpha^3)$ as $d\Omega_3^2 = (d\alpha^1)^2 + \sin^2\alpha^1\left((d\alpha^2)^2 + \sin^2\alpha^2\, (d\alpha^3)^2 \right)$. The Killing spinors in these coordinates take the form:

$$\epsilon = e^{\frac{i}{2}(3\tau+\phi+\psi)}\, e^{-i\frac{\varrho}{2}\Gamma_\varrho\gamma_*}\, e^{-i\frac{t}{2}\Gamma_t\gamma_*}\, e^{-\frac{\alpha^1}{2}\Gamma_{\alpha^1\varrho}}\, e^{-\frac{\alpha^2}{2}\Gamma_{\alpha^2\alpha^1}}\, e^{-\frac{\alpha^3}{2}\Gamma_{\alpha^3\alpha^2}}\, \eta\,, \tag{4.8}$$

where $\gamma_* \equiv \Gamma_t\, \Gamma_\varrho\, \Gamma_{\alpha^1\,\alpha^2\,\alpha^3}$ and η is a constant spinor that satisfies the same conditions as in (4.5).

Field	$R-\text{charge}$	number	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_{F_1}$	$\mathrm{U}(1)_{F_2}$
Y	$\frac{2}{3}\frac{x_3-x_1}{x_3}$	b	a	1	0
Z	$\frac{2}{3}\frac{x_3-x_2}{x_3}$	a	b	0	k
U_1	$\frac{2}{3}\frac{\alpha}{x_3}$	d	$-c$	0	l
U_2	$\frac{2}{3}\frac{\beta}{x_3}$	c	$-d$	-1	$-k-l$
V_1	$\frac{2}{3}\frac{2x_3+x_1-\beta}{x_3}$	$c-a$	$b-c$	0	$k+l$
V_2	$\frac{2}{3}\frac{2x_3+x_1-\alpha}{x_3}$	$b-c$	$c-a$	-1	$-l$

Table 1: Charges for bifundamental chiral fields in the quiver dual to $L^{a,b,c}$ [11].

5. Quiver theories for $L^{a,b,c}$ spaces

The $L^{a,b,c}$ superconformal field theories were first constructed in [10–12]. They are four dimensional quiver theories whose main features we would like to briefly remind. The gauge theory for $L^{a,b,c}$ has $N_g = a+b$ gauge groups and $N_f = a+3b$ bifundamental fields. The latter are summarized in table 1. There is a $\mathrm{U}(1)_F^2$ flavor symmetry that corresponds, in the gravity side, to the subgroup of isometries that leave invariant the Killing spinors. There is a certain ambiguity in the choice of flavor symmetries in the gauge theory side, as long as they can mix with the $\mathrm{U}(1)_B$ baryonic symmetry group. This fact is reflected in the appearance of two integers k and l in the $\mathrm{U}(1)_F^2$ charge assignments, whose only restriction is given by the identity $c\,k + b\,l = 1$ (here, it is assumed that b and c are coprime) [11].

The charge assignments in table 1 fulfill a number of nontrivial constraints. For example, all linear anomalies vanish, $\mathrm{Tr}\,\mathrm{U}(1)_B = \mathrm{Tr}\,\mathrm{U}(1)_{F_1} = \mathrm{Tr}\,\mathrm{U}(1)_{F_2} = 0$. The cubic t' Hooft anomaly, $\mathrm{Tr}\,\mathrm{U}(1)_B^3$, vanishes as well. The superpotential of the theory has three kind of terms; a quartic one,

$$Tr\,Y\,U_1\,Z\,U_2\,, \tag{5.1}$$

and two cubic terms,

$$Tr\,Y\,U_1\,V_2\,, \qquad Tr\,Y\,U_2\,V_1\,. \tag{5.2}$$

Their R-charge equals two and they are neutral with respect to the baryonic and flavor symmetries. The number of terms of each sort is uniquely fixed by the multiplicities of the fields to be, respectively, $2\,a$, $2\,(b-c)$ and $2\,(c-a)$ [11]. The total number of terms, then, equals $N_f - N_g$. In the $Y^{p,q}$ limit, the isometry of the space –thus the global flavor

symmetry– enhances, U^1 and U^2 (also V^1 and V^2) becoming a doublet under the enhanced SU(2) group. The superpotential reduces in this limit to the $Y^{p,q}$ expression [9]. More details about the $L^{a,b,c}$ superconformal gauge theories can be found in [10 – 12].

6. D3-branes on three-cycles

In this section we consider D3–brane probes wrapping three-cycles of $L^{a,b,c}$. These are pointlike objects from the gauge theory point of view, corresponding to dibaryons constructed from the different bifundamental fields of the quiver theory. There are other configurations of physical interest that we will not discuss in this letter. Though, we will briefly discuss their most salient features in section 9.

Given a D3–brane probe wrapping a supersymmetric three-cycle $\mathcal{C}$, the conformal dimension Δ of the corresponding dual operator is proportional to the volume of the wrapped three-cycle,

$$\Delta = \frac{\pi}{2} \frac{N}{L^3} \frac{\text{Vol}(\mathcal{C})}{\text{Vol}(L^{a,b,c})} \,. \tag{6.1}$$

Since the R-charge of a protected operator is related to its dimension by $R = \frac{2}{3}\Delta$, we can readily compute the R-charge of the dibaryon operators. On the other hand, the baryon number associated to the D3–brane probe wrapping $\mathcal{C}$ (in units of N) can be obtained as the integral over the cycle of the pullback of a $(2,1)$-form $\Omega_{2,1}$:

$$\mathcal{B}(\mathcal{C}) = \pm i \int_{\mathcal{C}} P[\Omega_{2,1}]_{\mathcal{C}} \,. \tag{6.2}$$

The explicit form of $\Omega_{2,1}$ is:

$$\Omega_{2,1} = \frac{K}{\rho^4} \left(\frac{dr}{r} + i\, e^5 \right) \wedge \left(e^1 \wedge e^2 - e^3 \wedge e^4 \right) , \tag{6.3}$$

where K is a constant that will be determine below. Armed with these expression, we can extract the relevant gauge theory information of the configurations under study.

6.1 U_1 dibaryons

Let us take the worldvolume coordinates for the D3-brane probe to be $\xi^\mu = (t, x, \psi, \tau)$, with $\theta = \theta_0$ and $\phi = \phi_0$ constant, and let us assume that the brane is located at a fixed point in AdS_5. The action of the kappa symmetry matrix on the Killing spinor reads

$$\Gamma_\kappa \epsilon = -\frac{i}{4! \sqrt{-\det g}} \epsilon^{\mu_1 \cdots \mu_4} \gamma_{\mu_1 \cdots \mu_4} \epsilon = -\frac{iL^4}{\sqrt{-\det g}} \left[a_5 \Gamma_{t5} + a_{135} \Gamma_{t135} \right] \epsilon \,, \tag{6.4}$$

where

$$a_5 = -i \frac{\cosh \varrho}{2\beta} \cos^2 \theta \,, \qquad a_{135} = -\frac{\cosh \varrho}{4\sqrt{\Delta_x}} \left(1 - \frac{x}{\beta} \right) \sqrt{\Delta_\theta} \sin(2\theta) \,. \tag{6.5}$$

Compatibility of (6.4) with the projections (4.5) demands $a_{135} = 0$. Since Δ_θ cannot vanish for positive α and β, this condition implies $\sin(2\theta) = 0$, i.e. $\theta = 0$ or $\pi/2$. Due to the fact that, for these configurations, the determinant of the induced metric is:

$$-\det g = \frac{L^8}{4} \left[\frac{\Delta_\theta \sin^2(2\theta)}{4\Delta_x} \left(1 - \frac{x}{\beta} \right)^2 + \frac{\cos^4 \theta}{\beta^2} \right] \cosh^2 \varrho \,, \tag{6.6}$$

we must discard the $\theta = \pi/2$ solution since the volume of the cycle would vanish in that case. Thus, the D3–brane probe is placed at $\theta = 0$ and the kappa symmetry condition $\Gamma_\kappa \epsilon = \epsilon$ reduces to the new projection:

$$\Gamma_{t5}\, \epsilon = -\epsilon \,, \tag{6.7}$$

which can only be imposed at the center of AdS_5. The corresponding configuration preserves four supersymmetries.

Given that the volume of $\mathcal{U}_1$ can be easily computed with the result

$$\mathrm{Vol}(\mathcal{U}_1\,) = \frac{\pi L^3}{\beta}\,(x_2 - x_1)\,\frac{\Delta\tau}{k}\,, \tag{6.8}$$

the corresponding value for the R-charge is:

$$R_{\mathcal{U}_1} = \frac{2}{3}\,\frac{\alpha}{\alpha + \beta - x_1 - x_2}\, N = \frac{2\alpha}{3x_3}\, N \,, \tag{6.9}$$

where we have used the second relation in (2.8). This result agrees with the value expected for the operator $\det(U_1)$ [11]. Let us now compute the baryon number associated to the D3–brane probe wrapping $\mathcal{U}_1$. For the $\mathcal{U}_1$ cycle, we get

$$\mathcal{B}(\mathcal{U}_1) = i \int_{\mathcal{U}_1} P[\,\Omega_{2,1}\,]_{\mathcal{U}_1} = -\frac{2\pi^2}{\alpha\beta}\,\frac{c}{a\,b}\,K \,, \tag{6.10}$$

where we have used the second identity in (2.10). From the field theory analysis [11] it is known that the baryon number of the U_1 field should be $-c$ (see table 1). We can use this result to fix the constant K to:

$$K = -\frac{\alpha\beta}{2\pi^2}\,a\,b\,. \tag{6.11}$$

Once it is fixed, formulas (6.2) and (6.3) allow us to compute the baryon number of any D3–brane probe wrapping a three-cycle.

6.2 U_2 dibaryons

Let us again locate the D3-brane probe at a fixed point in AdS_5 and take the following set of worldvolume coordinates $\xi^\mu = (t, x, \phi, \tau)$, with constant $\theta = \theta_0$ and $\psi = \psi_0$. The kappa symmetry matrix now acts on the Killing spinor as

$$\Gamma_\kappa \epsilon = -\frac{iL^4}{\sqrt{-\det g}}\,[\,b_5 \Gamma_{t5} + b_{135}\,\Gamma_{t135}\,]\,\epsilon \,, \tag{6.12}$$

where

$$b_5 = -i\frac{\cosh \varrho}{2\alpha}\,\sin^2\theta\,, \qquad b_{135} = \frac{\cosh \varrho}{4\sqrt{\Delta_x}}\left(1 - \frac{x}{\alpha}\right)\sqrt{\Delta_\theta}\,\sin(2\theta)\,. \tag{6.13}$$

The BPS condition is $b_{135} = 0$, which can only be satisfied if $\sin(2\theta) = 0$. We have to select now the solution $\theta = \frac{\pi}{2}$ if we want to have a non-zero volume for the cycle. The above condition defines the $\mathcal{U}_2$ cycle. The associated R-charge can be computed as above and reads:

$$R_{\mathcal{U}_2} = \frac{2\beta}{3x_3}\,N\,, \tag{6.14}$$

in precise agreement with the gauge theory result [11]. The baryon number reads

$$\mathcal{B}(\mathcal{U}_2) = i \int_{\mathcal{U}_2} P[\Omega_{2,1}]_{\mathcal{U}_2} = -c \, \frac{\beta}{\alpha} \, \frac{(\alpha - x_1)(\alpha - x_2)}{(\beta - x_1)(\beta - x_2)} \,, \qquad (6.15)$$

where we have used (6.11) and, after the third identity in (2.10), we get:

$$\mathcal{B}(\mathcal{U}_2) = -d = -(a + b - c) \,, \qquad (6.16)$$

In agreement with the field theory result [11] (see table 1). If we consider the case $a = p - q$, $b = p + q$ and $c = p$, which amounts to $Y^{p,q}$, a U(1) factor of the isometry group enhances to SU(2) and these dibaryons are constructed out of a doublet of bifundamental fields.

6.3 Y, Z dibaryons

We now take the following set of worldvolume coordinates $\xi^\mu = (t, \theta, \psi, \tau)$ and the embedding $x = x_0$ and $\psi' = \psi_0'$, where ψ_0' is a constant and $\psi' = 3\tau + \phi + \psi$ is the angle conjugated to the U(1)$_R$ charge. We implement this embedding in our coordinates by setting $\phi(\psi, \tau) = \psi_0' - 3\tau - \psi$. In this case

$$\Gamma_\kappa \epsilon = -\frac{iL^4}{\sqrt{-\det g}} \left[c_3 \Gamma_{t3} + c_5 \Gamma_{t5} + c_{135} \, \Gamma_{t135} \right] \epsilon \,, \qquad (6.17)$$

where

$$c_3 = 3i \, \frac{\rho \cosh \varrho}{2\alpha\beta} \, \sin(2\theta) \, \sqrt{\Delta_x} \,,$$

$$c_5 = i \, \frac{\cosh \varrho}{2\alpha\beta} \, \sin(2\theta) \, \left(3x^2 - 2(\alpha + \beta)x + \alpha\beta \right) \,,$$

$$c_{135} = \frac{\cosh \varrho}{\alpha\beta} \, \frac{\alpha \, \cos^2 \theta \, (1 - 3\sin^2 \theta) - \beta \, \sin^2 \theta \, (1 - 3\cos^2 \theta)}{\sqrt{\Delta_\theta}} \, \sqrt{\Delta_x} \,. \qquad (6.18)$$

The BPS conditions are, as before, $c_3 = c_{135} = 0$. Clearly these conditions are satisfied only if $\Delta_x = 0$, or, in other words, when

$$x = x_1 \,, x_2 \,. \qquad (6.19)$$

Notice that the value of ψ_0' is undetermined. The induced volume takes the form:

$$\sqrt{-\det g} \,\big|_{x=x_i} = \frac{L^4}{2\alpha\beta} \, \left| 3x_i^2 - 2(\alpha + \beta)x_i + \alpha\beta \right| \, \sin(2\theta) \, \cosh \varrho \,. \qquad (6.20)$$

As before, the compatibility with the AdS_5 SUSY requires that $\rho = 0$. Let us denote by $\mathcal{X}_i$ the cycle with $x = x_i$. We get that the volumes are given by:

$$\text{Vol}(\mathcal{X}_i) = \frac{\pi}{k \, \alpha \, \beta} \, \left| 3x_i^2 - 2(\alpha + \beta)x_i + \alpha\beta \right| \, \Delta\tau \, L^3 \,. \qquad (6.21)$$

From this result we get the corresponding values of the R-charges, namely:

$$R_{\mathcal{Y}} = \frac{2N}{3} \, \frac{x_3 - x_1}{x_3} \,, \qquad R_{\mathcal{Z}} = \frac{2N}{3} \, \frac{x_3 - x_2}{x_3} \,, \qquad (6.22)$$

where $\mathcal{Y} = \mathcal{X}_1$ and $\mathcal{Z} = \mathcal{X}_2$. Let us now compute the baryon number of these cycles. The pullback of the three-form $\Omega_{2,1}$ to the cycles with $x = x_i$ and $\psi' = \psi'_0$ is:

$$P[\Omega_{2,1}]_{x=x_i} = iK \frac{(3x_i^2 - 2(\alpha + \beta)x_i + \alpha\beta)}{2\alpha\beta} \frac{\sin(2\theta)}{\rho^4} d\theta \wedge d\psi \wedge d\tau \,, \qquad (6.23)$$

where K is the constant written in (6.11). We obtain:

$$\mathcal{B}(\mathcal{X}_i) = -i \int_{\mathcal{X}_i} P[\Omega_{2,1}]_{\mathcal{X}_i} = \frac{\pi}{k\,\alpha\beta} K \frac{3x_i^2 - 2(\alpha+\beta)x_i + \alpha\beta}{(\alpha - x_i)\,(\beta - x_i)} \Delta\tau \,. \qquad (6.24)$$

Taking into account the third identity in (2.10), we get:

$$\mathcal{B}(\mathcal{Y}) = a \,, \qquad \mathcal{B}(\mathcal{Z}) = b \,, \qquad (6.25)$$

as it should [11] (see table 1).

6.4 Generalized embeddings

In this subsection we show that there are generalized embeddings of D3–brane probes that can be written in terms of the local complex coordinates (3.3) as holomorphic embeddings or divisors of $\mathcal{C}L^{a,b,c}$. Let us consider, for example, (t, x, ψ, τ) as worldvolume coordinates and the ansatz

$$\theta = \theta(x, \psi) \,, \qquad \phi = \phi(x, \psi) \,. \qquad (6.26)$$

This ansatz is a natural generalization of the one used in section 6.1. The case where the worldvolume coordinate ψ is changed by ϕ, can be easily addressed by changing $\alpha \to \beta$ and $\theta \to \pi/2 - \theta$. Putting the D3-brane at the center of AdS_5, we get that the kappa symmetry condition is given by an expression as in (6.4)

$$\Gamma_\kappa \epsilon = -\frac{iL^4}{\sqrt{-\det g}} \left[a_5 \Gamma_{t5} + a_{135} \, \Gamma_{t135} \right] \epsilon \,, \qquad (6.27)$$

where a_5 and a_{135} are now given by:

$$a_5 = -\frac{i}{2} \left[\frac{\cos^2\theta}{\beta} + \frac{\sin^2\theta}{\alpha} \phi_\psi + \sin(2\theta) \left\{ \left(1 - \frac{x}{\beta}\right) \theta_x - \left(1 - \frac{x}{\alpha}\right) (\theta_x \phi_\psi - \theta_\psi \phi_x) \right\} \right] \,,$$

$$a_{135} = -\sqrt{\frac{\Delta_\theta}{\Delta_x}} \frac{\sin(2\theta)}{4} \left[1 - \frac{x}{\beta} - \left(1 - \frac{x}{\alpha}\right) \phi_\psi \right] + \sqrt{\frac{\Delta_x}{\Delta_\theta}} \left[\frac{\cos^2\theta}{\beta} \theta_x + \right.$$
$$\left. + \frac{\sin^2\theta}{\alpha} (\theta_x \phi_\psi - \theta_\psi \phi_x) \right] + \frac{i}{2} \left[\sqrt{\Delta_x \Delta_\theta} \frac{\sin(2\theta)}{\alpha\beta} \phi_x - \frac{\rho^2}{\sqrt{\Delta_x \Delta_\theta}} \theta_\psi \right] \,. \quad (6.28)$$

The BPS condition $a_{135} = 0$ reduces to the following pair of differential equations:

$$\frac{\cos^2\theta}{\beta} \theta_x + \frac{\sin^2\theta}{\alpha} (\theta_x \phi_\psi - \theta_\psi \phi_x) = \frac{\Delta_\theta}{\Delta_x} \left[1 - \frac{x}{\beta} - \left(1 - \frac{x}{\alpha}\right) \phi_\psi \right] \frac{\sin(2\theta)}{4} \,,$$

$$\rho^2 \theta_\psi = \frac{\Delta_x \Delta_\theta}{\alpha\beta} \sin(2\theta) \, \phi_x \,. \qquad (6.29)$$

The integral of the above equations can be simply written as:

$$z_2 = f(z_1),\tag{6.30}$$

where z_1 and z_2 are the local complex coordinates of $\mathcal{CL}^{a,b,c}$ and $f(z_1)$ is an arbitrary holomorphic function. Actually, if ξ^μ is an arbitrary worldvolume coordinate, one has:

$$\partial_{\xi^\mu} z_2 = f'(z_1)\,\partial_{\xi^\mu} z_1\,.\tag{6.31}$$

One can eliminate the function f in the above equation by considering the derivatives with respect to two worldvolume coordinates ξ^μ and ξ^ν. One gets:

$$\partial_{\xi^\mu} \log z_2\, \partial_{\xi^\nu}\, \log z_1 = \partial_{\xi^\nu}\, \log z_2\, \partial_{\xi^\mu}\, \log z_1\,.\tag{6.32}$$

Taking $\xi^\mu = x$ and $\xi^\nu = \psi$ in the previous equation and considering that the other coordinates θ and ϕ entering z_1 and z_2 depend on (x,ψ) (as in the ansatz (6.26)), one can prove that (6.32) is equivalent to the system of BPS equations (6.29).

We have checked that the Hamiltonian density of a static D3–brane probe of the kind discussed in this section satisfies a bound that is saturated when the BPS equations (6.29) hold. This comes from the fact that, from the point of view of the probes, these configurations can be regarded as BPS worldvolume solitons. We have also checked that these generalized embeddings are calibrated

$$P\Big[\frac{1}{2}\,J \wedge J\Big]_{\mathcal{D}} = \mathrm{Vol}(\mathcal{D})\,,\tag{6.33}$$

where $\mathrm{Vol}(\mathcal{D})$ is the volume form of the divisor $\mathcal{D}$, namely $\mathrm{Vol}(\mathcal{D}) = r^3\,dr \wedge \mathrm{Vol}(\mathcal{C})$. It is important to remind at this point that supersymmetry holds locally but it is not always true that a general embedding makes sense globally. We have seen examples of this feature in $Y^{p,q}$ [17].

7. D5-branes

Let us consider a D5-brane probe that creates a codimension one defect on the field theory. It represents a domain wall in the gauge theory side such that, when one crosses one of these objects, the gauge groups change and one passes from an $\mathcal{N} = 1$ superconformal field theory to a cascading theory with fractional branes. The setup for the supergravity dual of this cascading theory was proposed in [8].

We choose the following set of worldvolume coordinates: $\xi^\mu = (t, x^1, x^2, r, \theta, \phi)$, and we will adopt the ansatz $x = x(\theta, \phi)$, $\psi = \psi(\theta, \phi)$, $\tau = \tau(\theta, \phi)$ with x^3 constant. The kappa symmetry matrix acts on the spinor ϵ as:

$$\Gamma_\kappa \epsilon = \frac{i}{\sqrt{-\det g}}\frac{r^2}{L^2}\Gamma_{x^0 x^1 x^2 r}\gamma_{\theta\phi}\epsilon^* = \frac{i}{\sqrt{-\det g}}r^2\Gamma_{x^0 x^1 x^2 r}[b_I + b_{15}\Gamma_{15} + b_{35}\Gamma_{35} + b_{13}\Gamma_{13}]\epsilon^*,\tag{7.1}$$

where

$$b_I = \frac{i}{2} \left[\sin(2\theta) \left(1 - \frac{x}{\alpha} - \left(1 - \frac{x}{\beta} \right) \psi_\phi \right) - \frac{\sin^2 \theta}{\alpha} x_\theta + \frac{\cos^2 \theta}{\beta} \left(\psi_\theta \, x_\phi - \psi_\phi \, x_\theta \right) \right],$$

$$b_{15} = \frac{\rho}{\sqrt{\Delta_\theta}} \left[\left(1 - \frac{x}{\alpha} \right) \sin^2 \theta + \left(1 - \frac{x}{\beta} \right) \cos^2 \theta \, \psi_\phi + \tau_\phi \right] -$$
$$- \frac{i}{2} \sin(2\theta) \frac{\sqrt{\Delta_\theta}}{\rho} \left[\left(1 - \frac{x}{\alpha} \right) \tau_\theta + \left(1 - \frac{x}{\beta} \right) \psi_\theta \right] + \left(1 - \frac{x}{\beta} \right) \left(\tau_\phi \, \psi_\theta - \tau_\theta \, \psi_\phi \right) \right],$$

$$b_{35} = \frac{\sqrt{\Delta_x}}{\rho} \left[\frac{\alpha - \beta}{4\alpha\beta} \sin^2(2\theta) \, \psi_\theta - \frac{\sin^2 \theta}{\alpha} \tau_\theta + \frac{\cos^2 \theta}{\beta} \left(\tau_\phi \, \psi_\theta - \tau_\theta \, \psi_\phi \right) \right] +$$
$$+ \frac{i}{2} \frac{\rho}{\sqrt{\Delta_x}} \left[\left(1 - \frac{x}{\alpha} \right) \sin^2 \theta \, x_\theta - \left(1 - \frac{x}{\beta} \right) \cos^2 \theta \left(\psi_\theta \, x_\phi - \psi_\phi \, x_\theta \right) + \tau_\phi \, x_\theta - \tau_\theta \, x_\phi \right],$$

$$b_{13} = \frac{1}{4} \sqrt{\frac{\Delta_\theta}{\Delta_x}} \sin(2\theta) \left[\left(1 - \frac{x}{\alpha} \right) x_\theta + \left(1 - \frac{x}{\beta} \right) \left(\psi_\theta \, x_\phi - \psi_\phi \, x_\theta \right) \right] +$$
$$+ \sqrt{\frac{\Delta_x}{\Delta_\theta}} \left[\frac{\sin^2 \theta}{\alpha} + \frac{\cos^2 \theta}{\beta} \psi_\phi \right] + \frac{i}{2} \left[\frac{\rho^2}{\sqrt{\Delta_x \Delta_\theta}} x_\phi - \sqrt{\Delta_x \Delta_\theta} \frac{\sin(2\theta)}{\alpha\beta} \psi_\theta \right]. \tag{7.2}$$

The BPS conditions are $b_I = b_{15} = b_{35} = 0$. The imaginary part of b_{15} is zero if $\psi_\theta = \tau_\theta = 0$, i.e., $\psi = \psi(\phi)$, $\tau = \tau(\phi)$. Let us assume that this is the case and define the quantities n and m as:

$$\psi_\phi = n, \qquad \tau_\phi = m. \tag{7.3}$$

Clearly n and m are independent of the angle θ. Moreover, from the vanishing of the real part of b_{15} and of the imaginary part of b_{35} we get an algebraic equation for x, which can be solved as:

$$x = \alpha\beta \frac{\sin^2 \theta + n \cos^2 \theta + m}{\beta \sin^2 \theta + n\alpha \cos^2 \theta}. \tag{7.4}$$

On the other hand, when $\psi_\theta = 0$ and $\psi_\phi = n$ the vanishing of b_I is equivalent to the equation:

$$\left[\frac{\sin^2 \theta}{\alpha} + \frac{\cos^2 \theta}{\beta} n \right] x_\theta = \sin(2\theta) \left(1 - \frac{x}{\alpha} - \left(1 - \frac{x}{\beta} \right) n \right), \tag{7.5}$$

which is certainly satisfied by our function (7.4). For an embedding satisfying (7.3) and (7.4) one can check that $\sqrt{-\det g} = r^2 | b_{13} |$. Therefore, for these embeddings, Γ_κ acts on the Killing spinors as:

$$\Gamma_\kappa \, \epsilon = i e^{i\delta_{13}} \, \Gamma_{x^0 x^1 x^2 r} \, \Gamma_{13} \, \epsilon^*, \tag{7.6}$$

where δ_{13} is the phase of b_{13}. In order to implement correctly the kappa symmetry condition $\Gamma_\kappa \, \epsilon = \epsilon$, the phase δ_{13} must be constant along the worldvolume of the probe. By inspecting the form of the coefficient b_{13} in (7.2), one readily concludes that b_{13} must be real, which happens only when $x_\phi = 0$. Moreover, it follows from (7.4) that x is independent of ϕ only when n and m are constant. Thus, ψ and τ are linear functions of the angle ϕ, namely:

$$\psi = n\phi + \psi_0, \qquad \tau = m\phi + \tau_0, \tag{7.7}$$

where ψ_0 and τ_0 are constant. Notice that in these conditions the equation $\Gamma_\kappa \epsilon = \epsilon$ reduces to

$$i\, \Gamma_{x^0 x^1 x^2 r}\, \Gamma_{13}\, \epsilon^* = \epsilon \,. \tag{7.8}$$

Due to the presence of the complex conjugation, (7.8) is only consistent if the R-charge angle $\psi' = 3\tau + \phi + \psi$ is constant along the worldvolume (see the expression of ϵ in (4.4)). This in turn gives rise to an additional restriction to the possible supersymmetric embeddings. Indeed, the condition $3\tau + \phi + \psi = \psi'_0 =$ constant implies that the constants n and m satisfy

$$3m + n + 1 = 0 \,. \tag{7.9}$$

Thus, the possible supersymmetric embeddings of the D5-brane are labeled by a constant n and are given by:

$$\psi = n\phi + \psi_0 \,, \qquad \tau = -\frac{n+1}{3}\phi + \tau_0 \,,$$
$$x = \frac{\alpha\beta}{3} \frac{2 - n - 3(1-n)\cos^2\theta}{\beta + (n\alpha - \beta)\cos^2\theta} \,. \tag{7.10}$$

It can be now checked as in refs. [16, 17] that the projection (7.8) can be converted into a set of algebraic conditions on the constant spinors $\eta_\pm$ of (4.6). These conditions involve a projector which depends on the constant R-charge angle $\psi'_0 = 3\tau_0 + \psi_0$ and has four possible solutions. Therefore these embeddings are 1/8 supersymmetric.

The configuration obtained in this section can be also shown to saturate a Bogomol'nyi bound in the worldvolume theory of the D5–brane probes. This amounts to a point of view in which the solution is seen as a worldvolume soliton.

Other configurations of physical interest can be considered at this point. Most notably, we expect to find stable non-supersymmetric configurations of D5–branes wrapping three cycles of $L^{a,b,c}$. A similar solution where the D5–brane probe wraps the entire $L^{a,b,c}$ manifold, thus corresponding to the baryon vertex of the gauge theory, should also be found. We will not include the detailed analysis of these aspects in this article.

8. Spacetime filling D7-branes

Let us consider a D7–brane probe that fills the four Minkowski gauge theory directions while possibly extending along the holographic direction. These configurations are relevant to add flavor to the gauge theory. In particular, the study of fluctuations around them provides the meson spectrum. We start from the following set of worldvolume coordinates $\xi^\mu = (x^0, x^1, x^2, x^3, x, \psi, \theta, \phi)$ and the ansatz $r = r(x, \theta)$, $\tau = \tau(\psi, \phi)$. The kappa symmetry matrix in this case reduces to:

$$\Gamma_\kappa \epsilon = -i\, \frac{r^4}{L^4 \sqrt{-\det g}}\, \Gamma_{x^0 \cdots x^3}\, \gamma_{x\psi\theta\phi}\, \epsilon \,. \tag{8.1}$$

Let us assume that the Killing spinor ϵ satisfies the condition $\Gamma_* \epsilon = -\epsilon$, i.e. ϵ is of the form ϵ_- and, therefore, one has:

$$\Gamma_{x^0 \cdots x^3}\, \epsilon_- = i\epsilon_- \,, \tag{8.2}$$

which implies $\Gamma_{r5}\epsilon_- = i\epsilon_-$. Then:

$$\Gamma_\kappa\,\epsilon_- = \frac{r^4}{\sqrt{-\det g}}\,[\,d_I + d_{15}\,\Gamma_{15} + d_{35}\,\Gamma_{35} + d_{13}\Gamma_{13}\,]\,\epsilon_-\,. \qquad (8.3)$$

In order to express these coefficients in a compact form, let us define Λ_x and Λ_θ as:

$$\Lambda_x = -\frac{1}{2\Delta_x}\left[\,(\alpha - x)\,(\beta - x) + \alpha(\beta - x)\tau_\phi + \beta(\alpha - x)\tau_\psi\,\right]\,,$$
$$\Lambda_\theta = \frac{1}{\Delta_\theta}\left[(\alpha - \beta)\sin\theta\cos\theta + \alpha\,\cot\theta\,\tau_\phi - \beta\tan\theta\,\tau_\psi\right]\,. \qquad (8.4)$$

Then:

$$d_I = \frac{\sin\theta\cos\theta}{2\alpha\beta}\left[\rho^2 + \Delta_\theta\,\Lambda_\theta\,\frac{r_\theta}{r} + 4\,\Delta_x\,\Lambda_x\,\frac{r_x}{r}\right]\,,$$
$$d_{15} = i\rho\,\frac{\sin\theta\cos\theta}{2\alpha\beta}\,\sqrt{\Delta_\theta}\left[\frac{r_\theta}{r} - \Lambda_\theta\right]\,,$$
$$d_{35} = -\rho\,\frac{\sin\theta\cos\theta}{\alpha\beta}\,\sqrt{\Delta_x}\left[\frac{r_x}{r} - \Lambda_x\right]\,,$$
$$d_{13} = i\,\frac{\sin\theta\cos\theta}{\alpha\beta}\,\sqrt{\Delta_\theta\Delta_x}\left[\Lambda_x\,\frac{r_\theta}{r} - \Lambda_\theta\,\frac{r_x}{r}\right]\,. \qquad (8.5)$$

The BPS conditions are clearly $d_{15} = d_{35} = d_{13} = 0$. From the vanishing of d_{15} and d_{35} we get the following first-order equations:

$$\frac{r_\theta}{r} = \Lambda_\theta\,, \qquad \frac{r_x}{r} = \Lambda_x\,. \qquad (8.6)$$

Notice that $d_{13} = 0$ as a consequence of these equations. By looking at the explicit form of our ansatz and at the expression of Λ_θ and Λ_x in (8.4), one realizes that the only dependence on the angles ϕ and ψ in the first-order equations (8.6) comes from the partial derivatives of $\tau(\psi, \phi)$. For consistency these derivatives must be constant, i.e. $\tau_\psi = n_\psi$, $\tau_\phi = n_\phi$, where n_ψ and n_ϕ are constants. These equations can be trivially integrated:

$$\tau(\psi, \phi) = n_\psi\,\psi + n_\phi\,\phi + \tau_0\,. \qquad (8.7)$$

Notice that $\tau(\psi, \phi)$ relates angles whose periods are not congruent. Thus, the D7–brane spans a submanifold that is not, in general, a cycle. It is worth reminding that this is not a problem for flavor branes. If the BPS conditions (8.6) hold one can check that $r^4 d_I = \sqrt{-\det g}$ and, therefore, one has indeed that $\Gamma_\kappa\epsilon = \epsilon$ for any Killing spinor $\epsilon = \epsilon_-$, with ϵ_- as in (4.6). Thus these configurations preserve the four ordinary supersymmetries of the background.

In order to get the dependence of r on θ and x it is interesting to notice that, if $\tau(\psi, \phi)$ is given by (8.7), the integrals of Λ_θ and Λ_x turn out to be:

$$\int\Lambda_\theta\,d\theta = \log\left[\frac{(\sin\theta)^{n_\phi}(\cos\theta)^{n_\psi}}{\Delta_\theta^{\frac{n_\phi + n_\psi + 1}{2}}}\right]\,,$$
$$\int\Lambda_x\,dx = \log\left[\frac{[f_1(x)]^{\frac{n_\phi - n_\psi}{2}}}{\Delta_x^{\frac{1}{6}}[f_2(x)]^{\frac{n_\phi + n_\psi}{2} + \frac{1}{3}}}\right]\,, \qquad (8.8)$$

where $f_1(x)$ and $f_2(x)$ are the functions defined in (3.4). From this result it straightforward to obtain the general solution of $r(\theta, x)$:

$$r(\theta, x) = C \, \frac{(\sin\theta)^{n_\phi} \, (\cos\theta)^{n_\psi}}{\Delta_\theta^{\frac{n_\phi + n_\psi + 1}{2}}} \, \frac{[f_1(x)]^{\frac{n_\phi - n_\psi}{2}}}{\Delta_x^{\frac{1}{6}} \, [f_2(x)]^{\frac{n_\phi + n_\psi}{2} + \frac{1}{3}}} \, , \qquad (8.9)$$

where C is a constant. Notice that the function $r(x, \theta)$ diverges for some particular values of θ and x. This means that the probe always extends infinitely in the holographic direction. For particular values of n_ϕ and n_ψ there is a minimal value of the coordinate r, $r_\star$, which depends on the integration constant C. If one uses this probe as a flavor brane, $r_\star$ provides an energy scale that is naturally identified with the mass of the dynamical quarks added to the gauge theory.

It is finally interesting to write the embedding characterized by eqs. (8.7) and (8.9) in terms of the complex coordinates z_1, z_2 and z_3 defined in eq. (3.3). Indeed, one can check that this embedding can be simply written as:

$$z_1^{m_1} \, z_2^{m_2} \, z_3^{m_3} = \text{constant} \, , \qquad (8.10)$$

where $m_3 \neq 0$. The relation between the exponents m_i and the constants n_ψ and n_ϕ is the following:

$$\frac{m_1}{m_3} = \frac{3}{2} \left(n_\psi - n_\phi \right), \qquad \frac{m_2}{m_3} = -\frac{3}{2} \left(n_\psi + n_\phi \right) - 1 \, . \qquad (8.11)$$

By using the Dirac–Born–Infeld action of the D7–brane, it is again possible to show that there exists a bound for the energy which is saturated for BPS configurations.

9. Final remarks

In this letter we have worked out supersymmetric configurations involving D–brane probes in $AdS_5 \times L^{a,b,c}$. Our study focused on three kinds of branes, namely D3, D5 and D7. We have dealt with embeddings corresponding to dibaryons, defects and flavor branes in the gauge theory. For D3–branes wrapping three-cycles in $L^{a,b,c}$ we first reproduced all quantum numbers of the bifundamental chiral fields in the dual quiver theory. We also found a new class of supersymmetric embeddings of D3–branes in this background that we identified with a generic holomorphic embedding. The three-cycles wrapped by these D3–branes are calibrated. In the case of D5–branes, we found an embedding that corresponds to a codimension one defect. From the point of view of the D5–branes, it can be seen as a BPS saturated worldvolume soliton. We finally found a spacetime filling D7–brane probe configuration that can be seen to be holomorphically embedded in the Calabi–Yau, and is a suitable candidate to introduce flavor in the quiver theory.

Other interesting configurations have been considered following the lines of [17]. We would only list their main features:

Fat strings If we take a D3–brane with worldvolume coordinates (x^0, x^1, θ, ϕ) and consider an embedding of the form $x = x(\theta, \phi)$ and $\psi = \psi(\theta, \phi)$, with the remaining scalars constant, we see that there is no solution preserving kappa symmetry. However,

we have obtained a fat string solution by wrapping a probe D3–brane on a two-cycle, which is the same considered in section 7 for a D5–brane probe. This configuration is not supersymmetric but it is stable.

D5 on a three-cycle   We have found an embedding corresponding to D5–branes that wrap a three-cycle in $L^{a,b,c}$. They are codimension one in the gauge theory coordinates. These configurations happen to be non supersymmetric yet stable.

D5 on a two-cycle   We studied another embedding where a D5–brane probe wraps a two-cycle in $L^{a,b,c}$ while it extends along the radial coordinate. For this embedding, ϕ, ψ, x^3 and τ are held constant. This is a supersymmetric configuration. We also considered turning on a worldvolume flux in the case studied in section 7, and found that it can be done in a supersymmetric way. The flux in the worldvolume of the brane provides a bending of the profile x^3 of the wall, analogously to what happens in $T^{1,1}$ [16] and $Y^{p,q}$ [17].

Another spacetime filling D7   We considered a different spacetime filling D7–brane that extends infinitely in the radial direction and wraps a three-cycle holomorphically embedded in $L^{a,b,c}$ of the type studied in section 6.4. It preserves four supersymmetries.

D7 on $L^{a,b,c}$ We finally studied a D7–brane probe wrapping the entire $L^{a,b,c}$ space and extended along the radial coordinate. From the point of view of the gauge theory, this is a string-like configuration that preserves two supersymmetries.

It would be interesting to study in more detail the introduction of flavor in these theories and, in particular, to compute the corresponding meson spectra. These results exhaust the study of D–brane probes at the tip of toric Calabi–Yau cones on a base with topology $S^2 \times S^3$ initiated in [16, 17].

Acknowledgments

References

[1] E. Witten, *Baryons and branes in anti de Sitter space*, *JHEP* **07** (1998) 006 [hep-th/9805112].

[2] A. Karch and E. Katz, *Adding flavor to AdS/CFT*, *JHEP* **06** (2002) 043 [hep-th/0205236].

[3] S.S. Gubser, *Einstein manifolds and conformal field theories*, *Phys. Rev.* **D 59** (1999) 025006 [hep-th/9807164].

[4] I.R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, *Nucl. Phys.* **B 536** (1998) 199 [hep-th/9807080].

[5] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Sasaki-Einstein metrics on $S^2 \times S^3$*, *Adv. Theor. Math. Phys.* **8** (2004) 711 [hep-th/0403002].

[6] M. Cvetič, H. Lu, D.N. Page and C.N. Pope, *New Einstein-Sasaki spaces in five and higher dimensions*, *Phys. Rev. Lett.* **95** (2005) 071101 [hep-th/0504225]; *New Einstein-Sasaki and Einstein spaces from kerr-de Sitter*, hep-th/0505223.

[7] D. Martelli and J. Sparks, *Toric sasaki-einstein metrics on $S^2 \times S^3$*, *Phys. Lett.* **B 621** (2005) 208 [hep-th/0505027].

[8] D. Martelli and J. Sparks, *Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals*, *Commun. Math. Phys.* **262** (2006) 51 [hep-th/0411238].

[9] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, *An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals*, *JHEP* **06** (2005) 064 [hep-th/0411264].

[10] S. Benvenuti and M. Kruczenski, *From Sasaki-Einstein spaces to quivers via BPS geodesics: L(p,q|r)*, *JHEP* **04** (2006) 033 [hep-th/0505206].

[11] S. Franco et al., *Gauge theories from toric geometry and brane tilings*, *JHEP* **01** (2006) 128 [hep-th/0505211].

[12] A. Butti, D. Forcella and A. Zaffaroni, *The dual superconformal theory for L(p,q,r) manifolds*, *JHEP* **09** (2005) 018 [hep-th/0505220].

[13] M. Cederwall, A. von Gussich, B.E.W. Nilsson, P. Sundell and A. Westerberg, *The Dirichlet super-p-branes in ten-dimensional type-IIA and IIB supergravity*, *Nucl. Phys.* **B 490** (1997) 179 [hep-th/9611159];
E. Bergshoeff and P.K. Townsend, *Super D-branes*, *Nucl. Phys.* **B 490** (1997) 145 [hep-th/9611173];
M. Aganagic, C. Popescu and J.H. Schwarz, *D-brane actions with local kappa symmetry*, *Phys. Lett.* **B 393** (1997) 311 [hep-th/9610249]; *Gauge-invariant and gauge-fixed D-brane actions*, *Nucl. Phys.* **B 495** (1997) 99 [hep-th/9612080].

[14] K. Becker, M. Becker and A. Strominger, *Five-branes, membranes and nonperturbative string theory*, *Nucl. Phys.* **B 456** (1995) 130 [hep-th/9507158];
E. Bergshoeff, R. Kallosh, T. Ortin and G. Papadopoulos, *κ-symmetry, supersymmetry and intersecting branes*, *Nucl. Phys.* **B 502** (1997) 149 [hep-th/9705040];
E. Bergshoeff and P.K. Townsend, *Solitons on the supermembrane*, *JHEP* **05** (1999) 021 [hep-th/9904020].

[15] J.P. Gauntlett, J. Gomis and P.K. Townsend, *BPS bounds for worldvolume branes*, *JHEP* **01** (1998) 003 [hep-th/9711205].

[16] D. Arean, D.E. Crooks and A.V. Ramallo, *Supersymmetric probes on the conifold*, *JHEP* **11** (2004) 035 [hep-th/0408210].

[17] F. Canoura, J.D. Edelstein, L.A.P. Zayas, A.V. Ramallo and D. Vaman, *Supersymmetric branes on $AdS_5 \times Y^{(p,q)}$ and their field theory duals*, *JHEP* **03** (2006) 101 [hep-th/0512087].

[18] K. Sfetsos and D. Zoakos, *Supersymmetric solutions based on Y(p,q) and L(p,q,r)*, *Phys. Lett.* **B 625** (2005) 135 [hep-th/0507169].